THE RADIATION RESISTANCE OF CYLINDRICAL SHELLS EXHIBITING AXISYMMETRIC MODE SHAPES

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A study is made of the radiation resistance of long cylindrical shells in contact with an ideal compressible acoustic medium of infinite extent. The problem is formulated mathematically in terms of two descriptive differential equations: one for the cylindrical shell and the other for the acoustic medium. The solution to these equations is obtained by imposition of a boundary condition establishing velocity compatibility at the shell-fluid interface and by the requirement that the results satisfy the radiation condition in the limit at large distances from the surface of the shell. For convenience and generality, the results are obtained in terms of dimensionless series that are numerically evaluated for realistic ranges of the dimensionless parameters involved.

LIST OF SYMBOLS

a	mean-radius of cylinder
C_L	$C_L^2 = E/[\rho_s(1-\nu^2)]$
c_o	speed of sound in the acoustic medium
D	plate stiffness, $Eh^3/[12(1-\nu^2)]$
E	Young's Modulus
h	thickness of shell
i	$i^2 = -1$
k_r	radial wave number
m	integer
m_s	mass of shell material per unit of area
n	integer
p	acoustic pressure
q	applied load on cylinder surface per unit of area
r	radial coordinate in cylindrical coordinate system
R_{rad}	radiation resistance
t	time
u_r	radial fluid particle velocity
w	radial velocity of shell surface
W_r	radiated power
α	dimensionless axial coordinate
ϵ	$\epsilon^2 = D/a^4 m_s$
ν	Poisson's ratio
ρ_o	ambient density of the acoustic medium
ρ_s	density of shell material
ϕ	acoustic velocity potential
ω	frequency of forcing function
ω_r	ring frequency
ω_g	critical frequency

INTRODUCTION

The radiation resistance of cylindrical shells has not been studied in either great depth or with broad generality. In fact, the need to do so in relationship to acoustics problems has only recently materialized in conjunction with the development of statistical energy methods of vibration analysis [1, 2, 3, 4]. Morse [5], Junger [6, 7, 8], Bleich and Baron [9], Greenspon [10] and others have investigated the problem of cylindrical shell-acoustic media vibrational interaction from numerous points of view, but little work has been directed toward the determination of the radiation resistance. Recent work related to the radiation resistance of cylindrical shells has been done in terms of modal configuration studies [11] and numerical techniques [12, 13]. In contrast, this study employs a mathematical model consisting of simultaneous differential equations--one describing the motion of the shell and the other, the motion of the fluid. The solution of these equations is obtained by classical separation-of-variables techniques subject to a velocity compatibility boundary condition at the shell-fluid interface and to the radiation condition at large-distances from the shell surface. The results are utilized to compute the power radiated and the radiation resistance.

ANALYTICAL DEVELOPMENT

The problem of a long cylindrical shell in contact with an ideal compressible acoustic medium is examined. The descriptive differential equation for the shell is developed based on hypotheses that the length of the shell is large compared to the radius; the shell material is isotropic, elastic and obeys the thin shell equations of deformation; and the surface of the shell assumes a vibrational mode shape which can be mathematically described in terms of the axial coordinate alone.

Beginning with the general thin-shell differential equations [14] relating displacements to the like components of applied forces, the descriptive equation for the shell is developed by first noting the mathematical implication of the mode shape assumption; by combining the resulting equations into one equation in the radial displacement, w; and by considering the radial surface load to be composed of three components. The first component is the inertia or d'Alembert force; the second, the acoustic resisting force; and the third, the applied surface load, q, which is due to a source within the cylinder and will be considered to be a function of both time, t, and the axial coordinate, α. The force due to the presence of the acoustic medium will be expressed in terms of an acoustic velocity potential, Φ. The result is

$$\frac{\partial^2 w}{\partial t^2} + \frac{D}{m_s a^4}\left[\frac{\partial^4 w}{\partial \alpha^4} + \frac{Eha^2 w}{D}\right] = \frac{q(\alpha,t)}{m_s} + \frac{\rho_o}{m_s}\left[\frac{\partial \Phi}{\partial t}(r,\alpha,t)\right]_{r=a} . \qquad (1)$$

Equation (1) is an equation of motion for the shell in terms of the radial displacement, w, alone. This equation will be employed to completely describe the cylindrical shell for axisymmetric mode shapes, and will be solved simultaneously with the wave equation to obtain the desired expression for the radiation resistance.

The descriptive equation for the acoustic medium is the classical wave equation which in terms of the velocity potential, Φ, is

$$\nabla^2 \Phi = \frac{1}{c_o^2}\frac{\partial^2 \Phi}{\partial t^2} . \qquad (2)$$

At the shell-fluid interface, the boundary condition will represent the constraint that the radial velocity of the shell surface is equal to the radial velocity of the fluid particles in contact with the shell, and is expressed mathematically as

$$\frac{\partial w}{\partial t}(\alpha,t) = \frac{\partial \Phi}{\partial r}(r,\alpha,t)\Big|_{r=a} . \qquad (3)$$

At large distances from the surface of the shell as r approaches infinity, the physical implications of this boundary condition are that no reflections or other physical disturbances occur at the far boundaries of the acoustic medium. This constraint is termed the radiation condition and assures that the solutions to the wave equation represent outgoing waves. Mathematically,

$$\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial \Phi}{\partial r} - ik_r \Phi\right] = 0 \quad \text{for } k_r > 0, \qquad (4a)$$

and

$$\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial \Phi}{\partial r} - \bar{k}_r \Phi\right] = 0 \quad \text{for } \bar{k}_r = ik_r > 0. \qquad (4b)$$

The solutions of equation (2) subject to the boundary condition equations (4) are

$$\Phi(r,\alpha,t) = i\omega \sum_{n=1} A_n \sin\frac{n\pi a}{L}\alpha\, H_0^{(1)}(k_r r)e^{i\omega t} \qquad (5a)$$

for $k_r > 0$, and

$$\Phi(r,\alpha,t) = i\omega \sum_{n=1} A_n \sin\frac{n\pi a}{L}\alpha\, H_0^{(1)}(i\bar{k}_r r)e^{i\omega t} \qquad (5b)$$

for $\bar{k}_r = ik_r > 0$. The velocity potential is then determined except for the constant A_n. Equation (1) and the boundary condition equation (3) are the information required for its determination. Expanding the applied load on the surface of the cylinder as

$$q(\alpha,t) = \sum_{n=1} Q_n \sin\frac{n\pi a}{L}\alpha\, e^{i\omega t}, \tag{6}$$

and utilizing equation (3) yields an equation which can be simplified to the form

$$\{-[-k_r H_1^{(1)}(k_r a) - \frac{\rho_o}{m_s} H_0^{(1)}(k_r a)]\omega^2 + \epsilon^2[(\frac{n\pi a}{L})^4 + \frac{Eha^2}{D}](-k_r H_1^{(1)}(k_r a))\}A_n = \frac{Q_n}{m_s}. \tag{7}$$

However $H_m^{(1)}(x) = J_m(x) + iY_m(x)$, consequently substitution of this expression into equation (7) yields an equation which can be employed to determine A_n as is indicated below.

$$A_n = \frac{Q_n}{m_s\{[k_r J_1(k_r a) + \frac{\rho_o}{m_s} J_o(k_r a)]\omega^2 - \chi_n^2 k_r J_1(k_r a)\}}, \tag{8}$$

where

$$\chi_n^2 = \epsilon^2[(\frac{n\pi a}{L})^4 + \frac{Eha^2}{D}]. \tag{9}$$

Defining

$$\nu_n^2 = \frac{\chi_n^2}{1 + \frac{\rho_o}{m_s}\frac{J_o(k_r a)}{k_r J_1(k_r a)}}, \tag{10}$$

and rewriting equation (8) produces

$$A_n = \frac{Q_n \nu_n^2}{m_s k_r J_1(k_r a)\chi_n^2(\omega^2-\nu_n^2)}. \tag{11}$$

Consequently, the velocity potential is

$$\phi(r,\alpha,t) = \frac{i\omega}{m_s}\sum_{n=1}\frac{Q_n \nu_n^2}{\chi_n^2(\omega^2-\nu_n^2)}\sin\frac{n\pi a}{L}\alpha\,\frac{H_0^{(1)}(k_r r)e^{i\omega t}}{k_r J_1(k_r a)} \tag{12}$$

for $k_r > 0$, and for $\bar{k}_r = ik_r > 0$, similar manipulation yields

$$\phi(r,\alpha,t) = 0. \tag{13}$$

At this point, the equations for the velocity potential will be written in terms of a non-dimensional series to facilitate numerical evaluation. Defining

$$\eta = \frac{\omega a}{c_o}, \quad \xi = \frac{\nu_n a}{c_o}, \quad \text{and} \quad x_r = k_r a, \tag{14}$$

yields

$$\phi(r,\alpha,t) = \frac{i\omega Q_o}{m_s}(\frac{a^3}{c_o^2})\sum_{n=1} S_n(\alpha) H_0^{(1)}(k_r r)e^{i\omega t}, \tag{15}$$

for $k_r > 0$ and $\phi = 0$ for $\bar{k}_r = ik_r > 0$ with

$$S_n(\alpha) = \frac{\frac{Q_n}{Q_o}(\frac{\nu_n}{\chi_n})^2 \sin\frac{n\pi a}{L}\alpha}{[\eta^2-\xi^2]\, x_r J_1(x_r)}. \tag{16}$$

The radiation resistance will now be calculated. At large distances from the surface of the cylindrical shell, the acoustic pressure will be determined by noting that $p = \rho_o \frac{\partial\phi}{\partial t}$ and that the Hankel function can be asymptotically represented in terms of an exponential function. The acoustic pressure in the far-field is thus

$$p = \frac{-\omega^2\rho_o Q_o}{m_s}(\frac{a^3}{c_o^2})\sum_{n=1} S_n(\alpha)\sqrt{\frac{2}{\pi k_r r}}\, e^{i(\omega t + k_r r - \frac{\pi}{4})}. \tag{17}$$

The radial fluid particle velocity is obtained by noting that at large distances from the surface of the shell, the cylindrical wave front behavior approaches that of a plane wave front in any small increment of polar angle. Consequently the plane wave relationship between p and u_r will be employed: $u_r = p/\rho_o c_o$. The product of the real part of the acoustic pressure and the real part of the fluid velocity averaged over time is the intensity or radiated power per unit of acoustic field area,

$$I_\alpha = \frac{\rho_o\omega^4 Q_o^2}{2m_s^2}(\frac{a^6}{c_o^5})\sum_{n=1} S_n(\alpha)\sqrt{\frac{2}{\pi k_r r}}\sum_{m=1} S_m(\alpha)\sqrt{\frac{2}{\pi k_r r}}. \tag{18}$$

The total radiated power is

$$W_r = \int_0^{\frac{L}{a}} 2\pi r\cdot I_\alpha\cdot d\alpha.$$

Integrating and making use of orthogonality gives

$$W_r = \frac{\rho_o Q_o^2 a^2 L}{m_s^2 c_o} \left(\frac{\omega a}{c_o}\right)^4 \sum_{m=1} \left[\frac{\frac{Q_m}{Q_o}\left(\frac{\nu_m}{X_m}\right)^2}{x_r^{\frac{1}{2}}(\eta^2-\xi^2)}\right]^2 \left[\frac{1}{x_r J_1(x_r)}\right]^2 . \tag{19}$$

The mean-square surface velocity of the vibrating surface is now required, and will be calculated from Φ based on an equation similar to equation (3). The surface radial velocity, u_a, is determined. The real part of this quantity is squared to yield U_a^2, a quantity which is averaged over time and space to produce the mean square surface velocity, U^2.

$$U^2 = \left(\frac{Q_o a}{2m_s c_o}\right)^2 \left(\frac{\omega a}{c_o}\right)^2 \sum_{m=1} \left[\frac{\frac{Q_m}{Q_o}\left(\frac{\nu_m}{X_m}\right)^2}{\eta^2 - \xi^2}\right]^2 \left[1 + \frac{Y_1^2(x_r)}{J_1^2(x_r)}\right] . \tag{20}$$

The radiation resistance per characteristic length of the shell is thus $R_{rad} = 2W_r/U^2$ or

$$R_{rad} = 4\rho_o c_o L\left(\frac{\omega a}{c_o}\right)^2 \frac{\sum_{m=1} \left[\frac{\frac{Q_m}{Q_o}\left(\frac{\nu_m}{X_m}\right)^2}{\eta^2 - \xi^2}\right]^2 \left[\frac{1}{x_r^3 J_1^2(x_r)}\right]}{\sum_{m=1} \left[\frac{\frac{Q_m}{Q_o}\left(\frac{\nu_m}{X_m}\right)^2}{\eta^2 - \xi^2}\right]^2 \left[1 + \frac{Y_1^2(x_r)}{J_1^2(x_r)}\right]} , \tag{21}$$

for $k_r > 0$ and for $\bar{k}_r > 0$, $R_{rad} = 0$.

NUMERICAL EVALUATION

Numerical evaluation of the expression for the radiation resistance can be accomplished by writing x_r, ξ, and ν_n/X_n as dependent variables, functions of the independent variable η and other basic parameters which are related to the geometry of the shell and the physical properties of the shell material and fluid. Thus

$$x_r = \left[\eta^2 - \left(\frac{n\pi a}{L}\right)^2\right]^{\frac{1}{2}} , \tag{22}$$

$$\left(\frac{\nu_n}{X_n}\right)^2 = \frac{1}{1 + \left(\frac{\rho_o}{\rho_s}\right)\left(\frac{a}{L}\right)\left(\frac{h}{L}\right)^{-1} \frac{J_o(x_r)}{x_r J_1(x_r)}} , \tag{23}$$

and

$$\xi^2 = \left(\frac{\nu_n}{X_n}\right)^2 \left[\left(\frac{C_L}{c_o}\right)^2 \left\{\frac{(m\pi)^4}{12}\left(\frac{h}{L}\right)^2\left(\frac{a}{L}\right)^2 + (1-\nu^2)\right\}\right] . \tag{24}$$

Consequently all quantities in equation (21) are now represented in terms of the forcing frequency parameter, η; the shell geometry parameters, a/L, and h/L; and the material properties, ρ_o/ρ_s, C_L/c_o, and ν with $Q_m/Q_o = 1/n$.

The numerical analysis, per se, is a parameter study of the problem in terms of the previously mentioned parameters: the ranges of values for the shell geometry parameters and for η are approximately

$$10^{-2} \leq \eta \leq 10^3 ,$$

$$10^{-3} \leq a/L \leq 1 ,$$

and

$$10^{-5} \leq h/L \leq 10^{-1} .$$

A digital computer was employed to sum the dimensionless series which is finite for $k_r > 0$. The program incorporates this information and terminates summation for any value of n greater than $\eta/(\pi a/L)$. The program also terminates summation in the case of convergence to a stable value for each series.

The numerical results are shown in Figures 1 through 4. Figure 1 presents the dimensionless R_{rad} for small η: the acoustic medium is air. Because this results peaks at each resonance of the shell, an averaged or smoothed curve would be more useful in octave band analysis work. Hence the computer is employed to average the theoretical curve to obtain the averaged curve also shown in Figure 1. Figure 2 depicts the same information for an acoustic medium of water instead of air. Figure 3 is essentially a comparison of the results for small η for acoustic environments of either air or water. Finally in Figure 4, the averaged radiation resistance for a shell in contact with water is indicated for a wide range of η values.

DISCUSSION AND CONCLUSIONS

The results exhibit excellent agreement with the basic characteristics of previous work on the radiation resistance of short cylindrical shells [11]. Although the work reviewed in this report is theoretically applicable to long or mathematically infinite cylindrical shells, Figure 4, which indicates the averaged behavior of R_{rad} over a wide range of dimensionless frequency values, shows quite clearly the

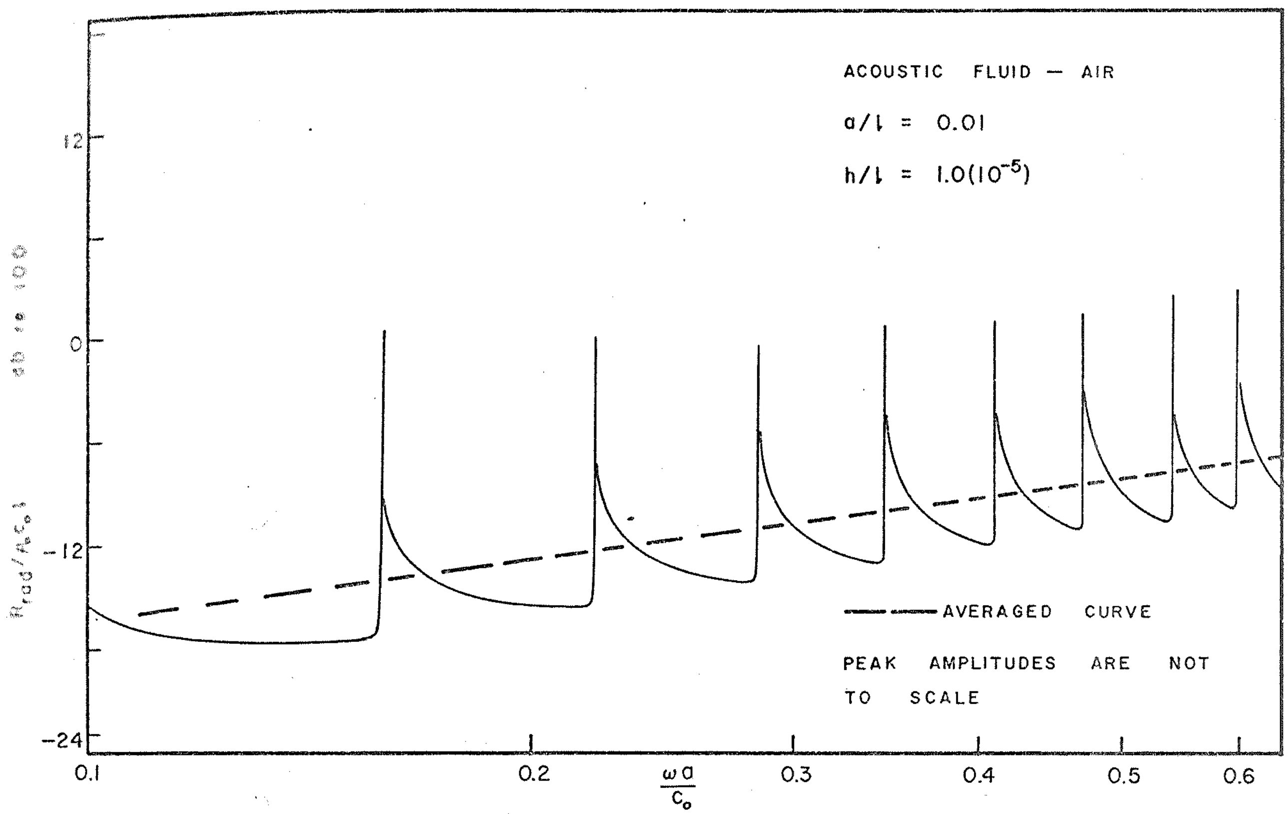

Figure 1 - Dimensionless radiation resistance vs $(\frac{\omega a}{c_o})$

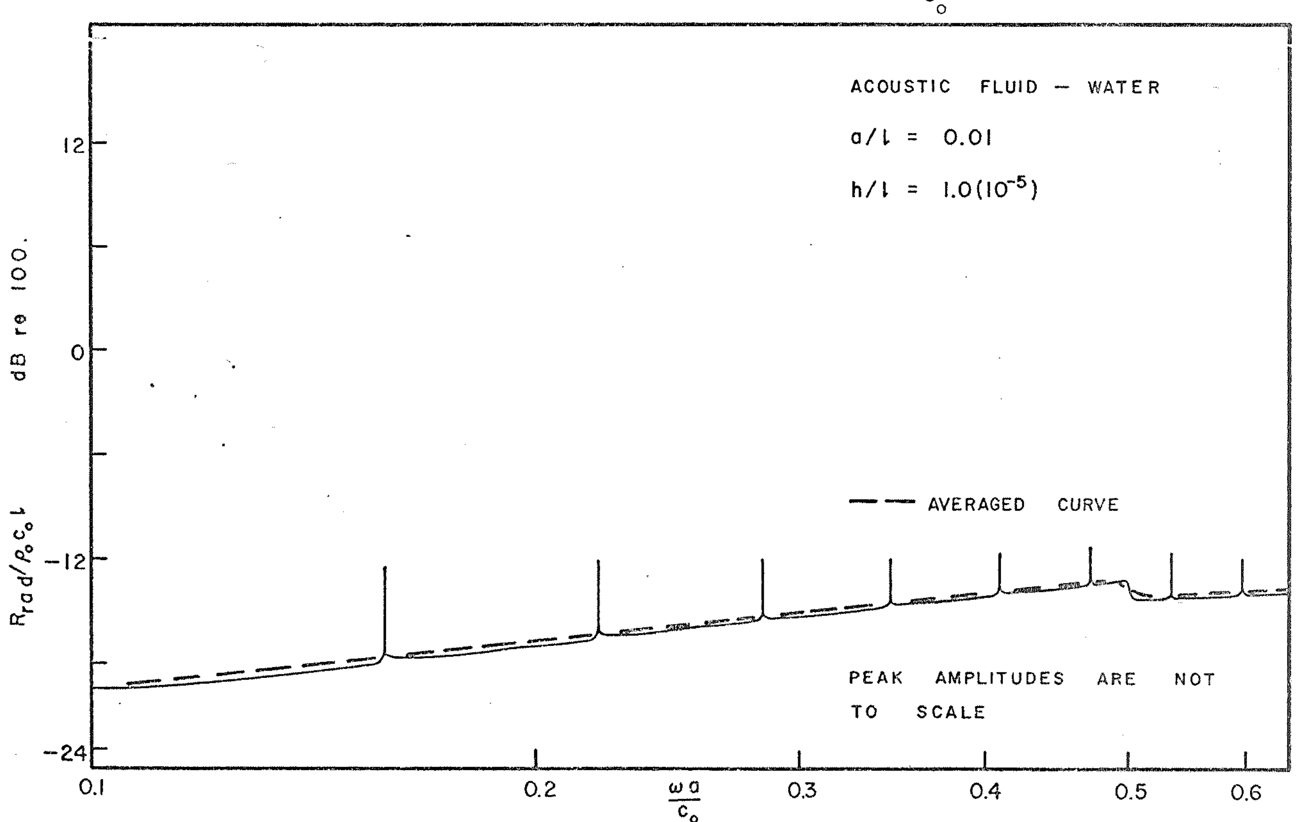

Figure 2 - Dimensionless radiation resistance vs $(\frac{\omega a}{c_o})$

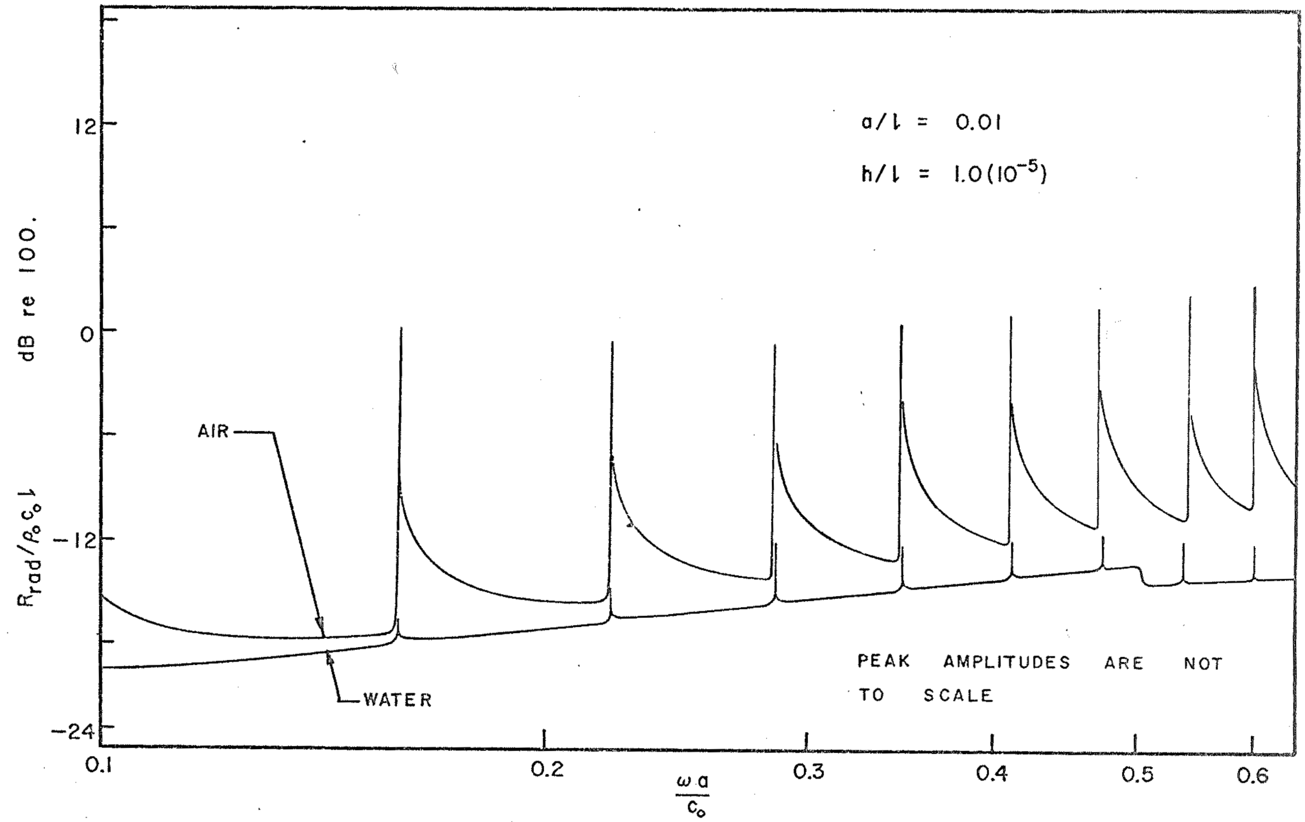

Figure 3 - Dimensionless radiation resistance vs $(\frac{\omega a}{c_o})$

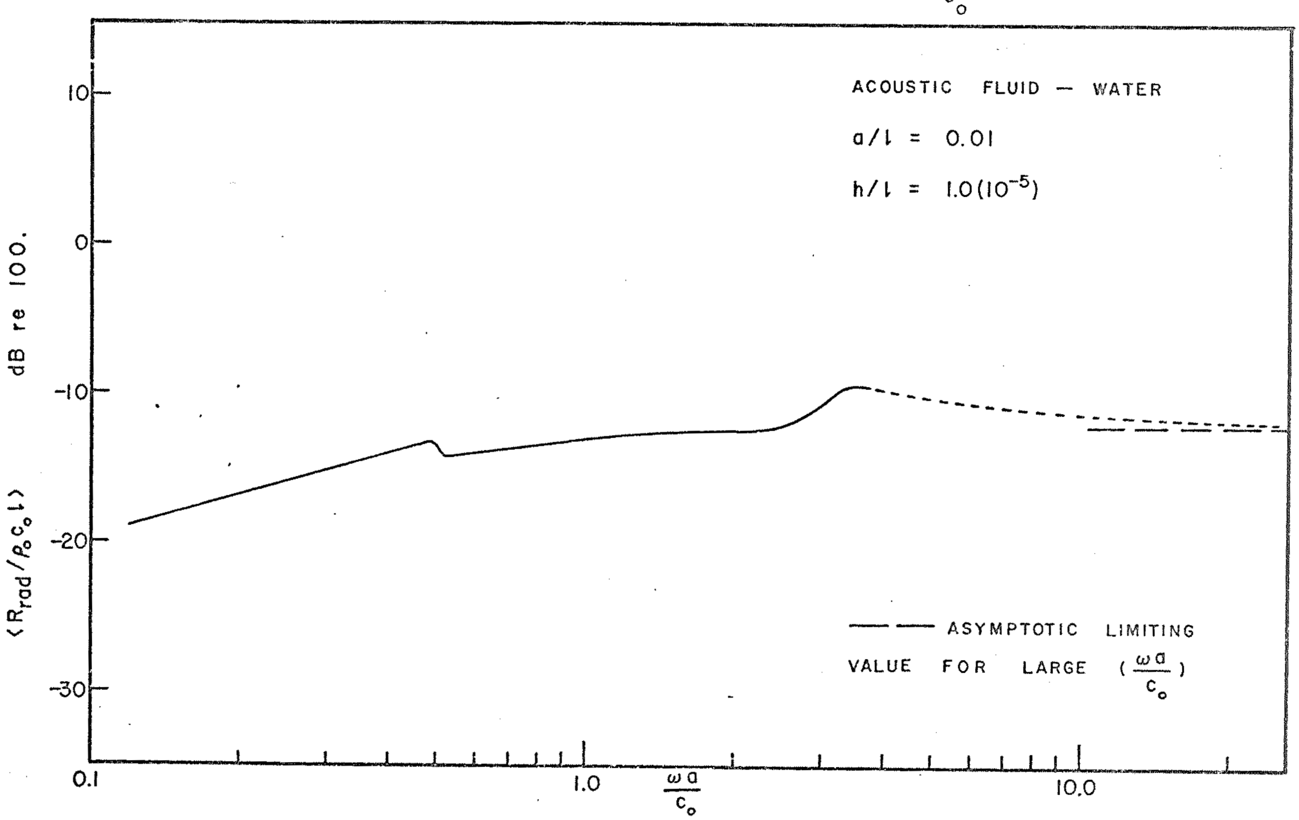

Figure 4 - Averaged dimensionless radiation resistance vs $(\frac{\omega a}{c_o})$

characteristics of previously published experimental data with regard to the two peaks in the low- to middle-frequency values and the asymptotic approach to the radiation resistance of a flat plate of equal area for large values of the dimensionless frequency parameter, η. The two peaks are identified as the ring frequency, ω_r, the frequency at which the longitudinal wave length in the cylinder material is equal to its circumference, and the critical frequency, ω_g, the frequency at which the flexural-wave speed in a flat plate of equivalent thickness is equal to the speed of sound in the surrounding acoustic medium, respectively. For large values of η, the results as indicated in Figure 4 oscillate between the dotted curve which represents the upper bound and the dashed curve which represents the asymptotic limit for values of the radiation resistance of the cylinder.

Due to the substantial differences in the shell and acoustic environment in the two cases--the theoretical work reported here applying to a long cylindrical shell in contact with an unbounded ideal fluid and the experimental work of Manning and Maidanik, for example, applying to a short, flanged cylinder in contact with a reverberant airspace--it is felt that the results of this work show good agreement with experimental studies. The decaying oscillation of the radiation resistance as η becomes large is an effect perhaps due to the combined influence of formulating the problem in terms of axisymmetric mode shapes of the cylinder and the anechoic acoustic environment. More study, both theoretical and experimental, is needed to clarify this point.

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DISCUSSION

Mr. Chaump (General Electric Co.): Was your source inside the cylinder a line source or a spherical source?

Mr. Runkle: It could be compared to a propagating wave.

Mr. Chaump: You were looking at longitudinal waves instead of radially outward waves?

Mr. Runkle: Yes.